

Deformations of Multiparameter Quantum $gl(N)$

C. Frønsdal

Department of Physics, UCLA, Los Angeles, California 90024-1547

A. Galindo

Departamento de Física Teórica
Universidad Complutense, 28040 Madrid, Spain

Abstract

Multiparameter quantum $gl(N)$ is not a rigid structure. This paper defines an essential deformation as one that cannot be interpreted in terms of a similarity transformation, nor as a perturbation of the parameters. All the equivalence classes of first order essential deformations are found, as well as a class of exact deformations. This work provides quantization of all the classical Lie bialgebra structures (constant r-matrices) found by Belavin and Drinfeld for $sl(n)$. A special case, that requires the Hecke parameter to be a cubic root of unity, stands out.

1. Introduction

Belavin and Drinfeld [1] classified the r -matrices (structures of coboundary Lie bialgebra) associated to simple Lie algebras, both finite and affine. A program of “quantization” of Lie algebras proposed by Drinfeld [2] would promote the classical structures to bialgebra deformations of enveloping algebras. Standard forms of such “quantum deformations” of the simple affine Lie algebras were obtained by Jimbo [3]. Here we are interested only in the finite case, of constant r - and R -matrices. So far there is no general classification of quantized Lie algebras.

Deformation theory applied at the classical point [4] is difficult, since the obstructions appear only in the second order. But a large family of exact quantum deformations of $gl(N)$ is known [5], with $1 + N(N - 1)/2$ parameters. These algebras are rigid (with respect to essential deformations) at generic points in parameter space, even to first order deformations, but essential deformations exist on algebraic surfaces of lower dimension, for $N > 2$. The determination of all the first order deformations, presented here, goes far towards a complete classification of all formal and/or exact deformations. All first order deformations are combinations of “elementary” deformations, and all the elementary deformations are exact. A family of exact deformations that contains the largest possible number of elementary ones is described. It is characterized by the fact that the dual Lie algebra has the largest possible semi-simple radical, which is the main reason why these new quantum groups are of interest in connection with integrable models of quantum field theories and classical statistical systems. We have in mind applications to new Toda field theories, statistical models and knot theory.

Added Notes. (1) This paper was distributed as a UCLA preprint in December, 1992. We have since become aware of a paper by Cremmer and Gervais [11] in which they describe an R -matrix that is substantially the same as the one we present in Section 9. (2) The Hecke condition that was imposed on the deformations considered in the original version of this paper is not essential; if the Hecke parameter $a \neq \pm 1$, then all the deformations that preserve the Yang-Baxter relation provide representations of the braid group factored through the Hecke algebra. The text has been modified to include this observation. (3) The cohomology that underlies the structure of the deformation space is not explained in this paper. However, the corresponding problem for Lie bialgebras has

been clarified in [12]. We hope to return to the question in a future publication.

2. Multiparameter Quantum $gl(N)$ and Quantum P-Algebras

We consider the free associative algebra \mathcal{F}_x generated by (x^i) , $i = \dots, N$, and the ideal \mathcal{F}_{x0} generated by

$$x^i x^j - q^{ij} x^j x^i, \quad i, j = 1, \dots, N, \quad (2.1)$$

in which the q 's are taken from a field K with characteristic 0, with

$$q^{ij} q^{ji} = 1, \quad q^{ii} = 1, \quad i, j = 1, \dots, N. \quad (2.2)$$

We call **quantum plane** the associative algebra $\mathcal{F}_x/\mathcal{F}_{x0}$; that is, the associative algebra generated by the x 's with relations

$$x^i x^j - q^{ij} x^j x^i = 0, \quad i, j = 1, \dots, N. \quad (2.3)$$

Similarly, the quantum anti-plane $\mathcal{F}_\theta/\mathcal{F}_{\theta0}$ is an associative algebra generated by N elements (θ^i) , $i = 1, \dots, N$, with relations

$$\theta^i \theta^j + r^{ij} \theta^j \theta^i = 0, \quad i, j = 1, \dots, N, \quad (2.4)$$

in which the r 's are parameters from K satisfying the same relations as the q 's, Eq. (2.2).

Let V denote the linear vector space over K spanned by the x 's (or by the θ 's).

Definition. A **generalized symmetry** is an element P of $\text{End}(V \otimes V)$ that satisfies the Hecke condition

$$(P - 1)(P + a) = 0, \quad (2.5)$$

for some $a \in K, a \neq -1, 0$. We shall say that the tensor $xx = (x^i x^j)$ is P -symmetric, and that the tensor $\theta\theta = (\theta^i \theta^j)$ is P -antisymmetric, if

$$xx(P - 1) = 0, \quad \theta\theta(P + a) = 0. \quad (2.3'-4')$$

Let P_{12} be the operator on $V \otimes V \otimes V$ that acts as P on the two first factors; the braid relation is

$$P_{12} P_{23} P_{12} = P_{23} P_{12} P_{23}. \quad (2.6)$$

Theorem 1. Given relations (2.3) and (2.4), with parameters q and r subject to the conditions (2.2), the following two statements are equivalent:

$$(a) \quad r^{ij} = aq^{ij}, \quad i < j, \quad i, j = 1, \dots, N; \quad (2.7)$$

(b) There exists P in $\text{End}(V \otimes V)$ satisfying (2.5) and (2.6), such that the relations (2.3-4) coincide respectively with Eqs. (2.3'-4'); it is unique up to a permutation of the basis.

Let P be a generalized symmetry of dimension N . Consider the algebra \mathcal{F}_x freely generated by (x^i) , $i = 1, \dots, N$, with the ideal \mathcal{F}_{x0} generated by

$$(xx(P-1))^{ij}, \quad i, j = 1, \dots, N; \quad (2.8)$$

and the algebra \mathcal{F}_θ freely generated by (θ^i) $i = 1, \dots, N$, with the ideal $\mathcal{F}_{\theta0}$ generated by

$$\theta\theta(P+a). \quad (2.9)$$

Let \mathcal{F} be the algebra generated by (x^i) and (θ^i) , $i = 1, \dots, N$ with relations

$$ax\theta = \theta xP. \quad (2.10)$$

This algebra contains \mathcal{F}_x and \mathcal{F}_θ as subalgebras and the ideals \mathcal{F}_{x0} and $\mathcal{F}_{\theta0}$ are thus canonically identified with subsets of \mathcal{F} .

Theorem 2. Suppose that the generalized symmetry P satisfies the braid relation. Let X denote the linear span of the x 's and Θ the linear span of the θ 's, then the statements

$$\mathcal{F}_{x0}\Theta = \Theta\mathcal{F}_{x0} \quad (2.11)$$

$$\mathcal{F}_{\theta0}X = X\mathcal{F}_{\theta0} \quad (2.12)$$

hold in \mathcal{F} . Conversely, if both (2.11) and (2.12) hold, then P satisfies the braid relation.

Proof. Eqs. (2.11), (2.12) are equivalent, respectively, to

$$(\text{braid})_{123}(P_{12} - 1) = 0, \quad (\text{braid})_{123}(P_{12} + a) = 0, \quad (2.13)$$

with

$$(\text{braid})_{123} := P_{12}P_{23}P_{12} - P_{23}P_{12}P_{23}. \quad (2.14)$$

Definition. Let $\langle q \rangle$ stand for a set of parameters (q^{ij}) , $i, j = 1, \dots, N$, satisfying $q^{ij}q^{ji} = 1$ and $q^{ii} = 1$; and a an additional parameter, all in the field K . Let $r^{ij} = aq^{ij}$ for $i < j$, $r^{ii} = 1$ and $r^{ji} = q^{ji}/a$ for $i < j$. The **standard quantum algebra** $A(\langle q \rangle, a)$ is generated by the x 's and the θ 's, with relations (2.3, 2.4) and (2.10). More generally, for any generalized symmetry P , the algebra $A(P)$ generated by the x 's and the θ 's, with relations (2.3'-4') and (2.10), will be called a **quantum P-algebra** if the conditions (2.11) and (2.12) hold.

The original aim of this work was to calculate the deformations of $A(\langle q \rangle, a)$ in the category of quantum P-algebras. Later it was discovered that the Hecke condition is inessential; it is enough to impose the braid relation, then the Hecke condition is satisfied automatically. Therefore we have, in fact, determined all the (equivalence classes of essential) deformations in the category of Yang-Baxter matrices. It is interesting that all the deformations lend themselves to an interpretation in terms of generalized symmetry. The quantum pseudogroup (in the sense of Woronowicz [6]) associated to P is the unital algebra generated by the matrix elements of an N -by- N matrix T , with relations

$$[P, T \otimes T] = 0. \quad (2.15)$$

It is the algebra of linear automorphisms of $A(P)$; that is, the set of mappings

$$(x, \theta, T) \rightarrow x \otimes T, \quad \theta \otimes T \quad (2.16)$$

that preserve the relations (2.3'-4', 2.10) of $A(P)$. It is related, via duality, to a quantum group in the sense of Drinfeld [2]. Twisted, quantum $gl(N)$ [5] corresponds to $\text{Aut}A(\langle q \rangle, a)$. The deformations of this quantum group are thus in 1-to-1 correspondence with the deformations of the standard quantum algebra $A(\langle q \rangle, a)$.

The next section defines the deformations that will be calculated in this paper. Then we begin, in Section 4, by calculating the first order deformations. It turns out that $A(\langle q \rangle, a)$ is rigid for parameters in general position. Interesting nontrivial deformations (even exact ones) do exist on certain algebraic surfaces in parameter space, for $N > 2$. The exact nontrivial deformations include the "esoteric" quantizations of $gl(N)$ reported elsewhere; they are described in Section 6. The existence of an unexpected special case that requires $a^3 = 1$ deserves some attention.

3. Deformations

Henceforth, P will denote the generalized symmetry associated with $A(\langle q \rangle, a)$. A **formal deformation** of $A(\langle q \rangle, a)$ is here a quantum $P(\epsilon)$ -algebra with $P(\epsilon)$ a formal power series in an indeterminate ϵ

$$P(\epsilon) = P + \epsilon P_1 + \epsilon^2 P_2 + \cdots, \quad (3.1)$$

that satisfies the Hecke condition with the parameter a independent of ϵ , and such that (2.11, 2.12) hold. In this case we shall say that $P(\epsilon)$ is a formal deformation of P . A deformation is **exact** if the series $P(\epsilon)$ has a nonvanishing radius of convergence.

If $P(\epsilon)$ is a formal deformation of P , then

$$P(\epsilon, 1) = P + \epsilon P_1 \quad (3.2)$$

is a first order deformation. More generally, a **first order deformation** is defined as a formal deformation except that one sets $\epsilon^2 = 0$. A first order deformation is not necessarily the first two terms of a formal deformation. For example, at the classical point, where a and all the q 's are equal to unity, the braid relation is moot in first order. (Recall that the condition that defines a formal deformation is in this case the classical Yang-Baxter relation, which is second order.) For this reason, the concept of a first order deformation is of no use at the classical point. In contrast with this, we shall find that, at general position in parameter space, $A(\langle q \rangle, a)$ is rigid with respect to first order deformations, which implies rigidity under formal deformations. The case $a = 1$, with the q 's in general position, is in this respect intermediate; it is best treated separately.

Two types of deformations (and combinations of them) will be considered trivial. A linear transformation, with coefficients in $K[\epsilon]$,

$$x^i \rightarrow x^i + \epsilon x^j A_j^i + \cdots, \quad \theta^i \rightarrow \theta^i + \epsilon \theta^j A_j^i + \cdots, \quad (3.3)$$

induces a trivial, formal deformation of P . A variation of the q 's

$$q^{ij} \rightarrow q^{ij} + \epsilon \delta q^{ij} + \cdots \quad (3.4)$$

will also be considered trivial. A deformation that is not trivial is called essential. Two first order deformations P_1 and P'_1 are equivalent if the difference is trivial; that is, induced

by transformations of the type (3.3), (3.4). We shall classify the equivalence classes of first order deformations. The following result is deduced from an examination of the case $N = 2$ in the next section.

Theorem 3. If $a \neq 1$, then each equivalence class of first order deformations contains a unique representative with the property that $(P_1)_{kl}^{ij} = 0$ for every index set i, j, k, l that contains no more than two different numbers.

The first order deformation of P induced by (3.3) is

$$P_1 = PZ - ZP, \quad Z := A \otimes 1 + 1 \otimes A, \quad (3.5)$$

or more explicitly

$$(P_1)_{kl}^{ij} = a(\hat{q}^{lk} - \hat{q}^{ij})Z_{kl}^{ji} + (1 - a)[(k < l) - (i < j)]Z_{kl}^{ij}, \quad (3.6)$$

where

$$\hat{q}^{ij} := \begin{cases} q^{ij} & \text{if } i < j, \\ q^{ij}/a & \text{if } i \geq j, \end{cases} \quad (i < j) := \begin{cases} 1 & \text{if } i < j, \\ 0 & \text{otherwise.} \end{cases} \quad (3.7)$$

Preservation of the Hecke condition (2.5) under first order deformations is equivalent to requiring that

$$\begin{aligned} q^{ij}(P_1)_{lk}^{ji} + \hat{q}^{kl}(P_1)_{kl}^{ij} &= 0 \\ (1 - a)q^{ij}(P_1)_{lk}^{ji} &= (a - 1)\hat{q}^{kl}(P_1)_{kl}^{ij} = (P_1)_{lk}^{ij} + aq^{ij}\hat{q}^{kl}(P_1)_{kl}^{ji}, \quad i \leq j, \quad k \leq l. \end{aligned} \quad (3.8)$$

The main difficulty is to extract the conditions on P_1 imposed by the braid relation. We found that the strategy made available by Theorem 2 simplifies this task. Both conditions, (2.11) and (2.12), must be invoked. We leave out the details.

4. The Case $N = 2$

The components of P_1 come in three disjoint sets. We comment first on the generic case, a and $q := q^{12}$ in general position.

(1) Since $a \neq -1$, the Hecke condition requires

$$(P_1)_{11}^{11} = (P_1)_{22}^{11} = (P_1)_{11}^{22} = (P_1)_{22}^{22} = 0. \quad (4.1)$$

(2) The components $(P_1)_{12}^{12}, (P_1)_{12}^{21}, (P_1)_{21}^{12}, (P_1)_{21}^{21}$ are unaffected by the transformations (3.3). The implications of the Hecke condition and the braid relation (to order ϵ) are

$$(P_1)_{12}^{12} + (P_1)_{21}^{21} = 0, \quad (a - 1)(P_1)_{21}^{21} = 0, \quad (P_1)_{21}^{12} + aq^2(P_1)_{12}^{21} = 0. \quad (4.2)$$

This allows for a one-dimensional space of deformations that amounts to a variation of q ; hence it is trivial.

(3) The remaining coefficients are affected by the transformations (3.3). There are eight of them, of which 4 remain independent after imposition of the Hecke condition. Eq. (2.11) imposes two additional constraints, namely

$$\begin{aligned} (q - 1)(P_1)_{12}^{11} + (aq - 1)(P_1)_{22}^{21} &= 0, \\ (q - 1)(P_1)_{21}^{22} + (q - 1/a)(P_1)_{11}^{12} &= 0. \end{aligned} \quad (4.3)$$

This leaves two free parameters; the deformations are induced by transformations of the type (3.3), so they are trivial. No additional conditions from (2.12).

The exceptional cases are as follows. If $a = q = 1$ we get nothing at all from the braid relation, to first order. The classical Yang-Baxter relation appears in the second order. The only other special case is $a = 1, q \neq 1$. We now have two additional free parameters, say $(P_1)_{21}^{12}$ and $(P_1)_{12}^{12}$. The former is interpreted as a variation of q ; the latter is nontrivial. No additional restrictions from imposing (2.12).

The space of nontrivial, first order deformations is thus one-dimensional in the exceptional case when $a = 1, q \neq 1$. But this first order deformation is not the first order part of a formal (power series) deformation. **The space of essential, formal deformations of $A(\langle q \rangle, a)$ is empty in the case $N = 2$, unless $a = q = 1$.**

5. The Case $N = 3$

We shall assume, from here onwards, that $a^2 \neq 1$. In view of Theorem 3, we may “fix the gauge” by setting $(P_1)_{kl}^{ij} = 0$ unless all three of the values 1, 2, 3 appear among the indices.

We find that there are two very different cases:

(i) When the repeated index k lies between the other two, $i < k < j$, then the complete set of conditions is

$$\begin{aligned} (P_1)_{ij}^{kk} = -aq^{ji}(P_1)_{ji}^{kk} \neq 0 &\Rightarrow q^{ik} = q^{kj} \quad \text{and} \quad q^{ij} = (q^{ij})^2, \\ (P_1)_{kk}^{ij} = -q^{ij}(P_1)_{kk}^{ji} \neq 0 &\Rightarrow q^{ik} = q^{kj} \quad \text{and} \quad q^{ij} = a(q^{ik})^2. \end{aligned} \quad (5.1)$$

(ii) In the other cases one finds:

$$(P_1)_{ij}^{kk} = -aq^{ji}(P_1)_{ji}^{kk} \neq 0 \Rightarrow \begin{cases} q^{ik} = a^2 q^{kj}, q^{ij} = (q^{kj})^2 & \text{if } k < i < j, \\ q^{ik} = a^{-2} q^{kj}, q^{ij} = (q^{ik})^2 & \text{if } i < j < k, \end{cases} \quad (5.2)$$

$$(P_1)_{kk}^{ij} = -q^{ij}(P_1)_{kk}^{ji} \neq 0 \Rightarrow \begin{cases} q^{ik} = q^{kj}, q^{ij} = a(q^{kj})^2 & \text{if } k < i < j, \\ q^{ik} = q^{kj}, q^{ij} = a(q^{ik})^2 & \text{if } i < j < k, \end{cases} \quad (5.3)$$

and, in addition, very unexpectedly, in this case

$$a^3 = 1. \quad (5.4)$$

6. The Case $N = 4$

The deformations found for $N = 3$ give rise to a class of deformations for general N that may be called class-3 deformations. [The negative result for $N = 2$ means that there are no class-2 deformations.] These class-3 deformations will be discussed later; here we investigate those of class 4; that is, those that involve $(P_1)_{kl}^{ij}$'s with four different index values. They are all essential (first order) deformations.

The Hecke condition and the braid relation connect P_1 's with the same set (i, j) of upper indices and the same set (k, l) of lower indices to each other. There are six sets, distinguished by the relative order of the four indices. We fix $i < j$ and $k < l$. The result is that deformations satisfying (2.11), (2.12) exist in 2 of the 6 cases, the conditions being, for $a^2 \neq 1$:

$$\begin{aligned} k < i < j < l: \quad x = y = 1/a, \quad u = v = 1, \quad \delta P_{kl}^{ij} &= 0, \\ i < k < l < j: \quad x = y = a, \quad u = v = 1, \quad \delta P_{kl}^{ij} &= 0, \end{aligned} \quad (6.1)$$

with

$$x := q^{ij}q^{jk}q^{jl}, y := q^{ij}q^{ki}q^{li}, u := q^{kl}q^{lj}q^{li}, v := q^{kl}q^{ik}q^{jk}, xu = yv. \quad (6.2)$$

The space of class-4 deformations is thus at most 1-dimensional (for $N = 4$).

7. First Order Deformations in the General Case

When $N > 3$ there are class-3 deformations involving $(P_1)_{ij}^{kk}$ and/or $(P_1)_{kk}^{ij}$, for each triple i, j, k of unequal numbers in $1, \dots, N$. But there are now additional conditions that must be satisfied by the additional q 's, and i, j, k must be nearest neighbours.

When $N > 4$ there are class-4 deformations satisfying (6.1). But the braid relation imposes additional conditions on the additional q 's, and the position of i, j, k, l inside the set $1, 2 \dots N$ is restricted so that either $k + 1 = i < j = l - 1$, or else $i + 1 = k < l = j - 1$.

Definition. An elementary first order deformation is one in which some $(P_1)_{kl}^{ij}$ is non-zero for just one unordered pair i, j and just one unordered pair k, l .

Theorem 4. Suppose $a \neq \pm 1, 0$. There are two series of elementary, first order, essential deformations. The “principal series” is described first. Let i, j be any index pair with, either (case 1) $k + 1 = i \leq j = l - 1$, or else (case 2) $i + 1 = k \leq l = j - 1$. Let $P_1 = 0$, except that

$$(P_1)_{lk}^{ij} = -a\hat{q}^{ij}q^{kl}(P_1)_{kl}^{ji} \neq 0. \quad (7.1)$$

This defines an elementary deformation if and only if the parameters satisfy the conditions

$$q^{im}q^{jm}q^{mk}q^{ml} = \begin{cases} a^x, & \text{case 1,} \\ a^{-x}, & \text{case 2;} \end{cases} \quad x = \delta_m^i - \delta_m^j, \quad m = 1, 2 \dots N. \quad (7.2)$$

The “exceptional series” of elementary, first order deformations exists only if $a^3 = 1$. Let i, j, k be neighbors in the natural numbers, with $i + 1 = j$. Let $P_1 = 0$, except that either $(P_1)_{kk}^{ij} = -aq^{ij}(P_1)_{kk}^{ji} \neq 0$, or else $(P_1)_{ij}^{kk} = -q^{ij}(P_1)_{ji}^{kk} \neq 0$, but not both. This defines an elementary deformation if and only if the parameters satisfy

$$\begin{aligned} (P_1)_{kk}^{ij} \neq 0 : \quad & (q^{km})^2 q^{mj} q^{mi} = a^x, \quad x = \delta_{mi} - \delta_{mj}; \\ (P_1)_{ij}^{kk} \neq 0 : \quad & (q^{km})^2 q^{mj} q^{mi} = a^x, \quad x = \pm(\delta_{mk} - \delta_{mi}). \end{aligned} \quad (7.3)$$

The two signs in the last line apply when $k = i - 1, k = j + 1$, respectively. There are no other first order, elementary deformations. The elementary deformations are formal and exact.

8. The Classical Limit

All the results obtained here, for twisted quantum $gl(N)$, have direct application to twisted, quantum $sl(N)$. The connection between these two was explained by Schirmacher in [5] and is discussed also in [10]. In this section we shall take the classical limit and confront our results for $gl(N)$ with the classification, by Belavin and Drinfeld [1], of the classical r-matrices for $sl(N)$. Strictly, this is possible only under additional conditions on the parameters, namely

$$\prod_i q^{ij} a^j = a^{(N+1)/2}, \quad (8.1)$$

we therefore assume that these relations hold, although they do not interfere directly with the following calculations.

The deformed quantum P-algebras of the principal series are semiclassical. The classical r-matrix is defined by expanding the parameters

$$a = 1 + h, \quad q^{ij} = 1 + hp^{ij}, \quad i < j, \quad (8.2)$$

and the R-matrix,

$$R_{lk}^{ij} := P_{kl}^{ij} + \epsilon(P_1)_{kl}^{ij}, \quad (8.3)$$

in powers of h ,

$$R = 1 - hr_\epsilon + O(h^2), \quad r_\epsilon = r + \epsilon\delta r. \quad (8.4)$$

Here r is the r-matrix for twisted (= multiparameter) $gl(N)$,

$$r = \sum_{i < j} M_j^i \otimes M_i^j + r_0, \quad (8.5)$$

with

$$r_0 := \sum_{i < j} (p^{ij} M_j^j \otimes M_i^i - (1 + p^{ij}) M_i^i \otimes M_j^j), \quad (8.6)$$

and the perturbation is

$$h\delta r_{kl}^{ij} = (P_1)_{lk}^{ij}, \quad \text{or} \quad h\delta r = \sum_{i,j,k,l} (P_1)_{lk}^{ij} M_k^i \otimes M_l^j. \quad (8.7)$$

Here M_k^j is the matrix with the unit in row- k , column- j , all the rest zero.

We examine the classical limit of an elementary, first-order deformation. Fix the notation as in Theorem 4, the expression for δr is, up to a constant,

$$\delta r = M_k^i \otimes M_l^j - M_l^j \otimes M_k^i. \quad (8.8)$$

The diagonal matrices (M_i^i) , $i = 1, \dots, N$, will be taken as a basis for a “Cartan subalgebra of $gl(N)$ ”. The upper triangular matrices form the subspace of positive roots and the matrices M_i^j with $i - j = \pm 1$ are the simple roots. [We have abused the notation by extending the notion of roots from $sl(n)$ to $gl(n)$ and by introducing both positive and negative “simple” roots.] The conditions on the indices that are spelled out in Theorem 4 insure that all the roots appearing in (8.8) are simple, and that a positive root is paired with a negative root and *vice versa*. Let us find out what is the meaning of the restriction (7.2) on the parameters.

Let $\Gamma_1(\Gamma_2)$ be the root space spanned by $M_k^i(M_j^l)$ and $\tau : \Gamma_1 \rightarrow \Gamma_2$ the mapping defined by $\tau(M_k^i) = M_j^l$. Consider the equation

$$(\tau\alpha \otimes 1 + 1 \otimes \alpha)r_0 = 0, \quad (8.9)$$

where $\alpha = M_k^i$ and $(\alpha \otimes 1)H \otimes H' = \alpha(H)H'$ and $(1 \otimes \alpha)H \otimes H' = \alpha(H')H$, H and H' in the diagonal subalgebra of $gl(N)$. We find that this equation is equivalent to

$$p^{lm} + p^{km} + p^{mi} + p^{mj} = \delta_m^j - \delta_m^i, \quad (8.10)$$

This equation is precisely the first order analog of Eq. (7.2), while (8.9) is a condition (Eq. (6.7)) of Belavin and Drinfeld [1], applied to the case of the elementary deformation (8.8). We have thus established that the conditions (7.2) on the parameters lift the invariance condition of ref. [1] to the quantum algebra.

9. A Family of Exact Deformations.

The elementary deformations are exact. Combinations of elementary deformations are not always exact and corrections of order ϵ^2 are needed. The interesting question is whether, for any first order deformation, the requirements on the next order correction has a solution (and so on). The case $N = 2, a = 1$, shows that this is not always the case. It seems likely that the question has a cohomological formulation.

The dual Lie algebra of a standard, simple Lie bialgebra, defined by the simplest classical r-matrix, is solvable. Since, in the application to solvable field theories, this dual algebra is the algebra of principal dynamical variables, the nature of the physical applications is strongly limited. The twisted forms of quantum $gl(N)$ do not offer anything new in this respect, and indeed, multiparameter quantum $gl(N)$ has been stigmatized as a mere “gauge transformation” of the standard version [7]. Esoteric quantum $gl(N)$ is quite different. We describe here a special case that may represent the strongest departure from a solvable dual. The special cases $gl(3)$ and $gl(5)$ were described in some detail in a preprint [8], and $gl(2n-1)$ in a conference report [9]. Here is “esoteric quantum $gl(2n-1)$ ” with some additional information.

We begin with multiparameter $gl(2n-1)$, with the special values of the parameters

$$\begin{aligned} q^{ij} &= 1/q, & i < j, & \quad i+j \neq 2n, & \quad (p^{ij} = -1/2) \\ q^{ij} &= 1/q^2, & i < j, & \quad i+j = 2n, & \quad (p^{ij} = -1) \end{aligned} \quad (9.1)$$

and $a = q^2$, with the R-matrix

$$\begin{aligned} R_0 &= \sum_{i=1}^{2n-1} M_i^i \otimes M_i^i + \sum_{i < j < 2n} (1 - q^2) M_j^i \otimes M_i^j + \sum_{i \neq j, i+j \neq 2n} q M_i^i \otimes M_j^j \\ &\quad + \sum_{i < n} M_i^i \otimes M_{i'}^{i'} + \sum_{i < n} q^2 M_{i'}^{i'} \otimes M_i^i, \end{aligned} \quad (9.2)$$

The R-matrix for the deformation that we have called esoteric quantum $gl(2n-1)$ is

$$\begin{aligned} R &= R_0 + R_1, \\ R_1 &= \sum_{k < i < n} (\mu'_i M_i^n \otimes M_{i'}^n + \mu_i M_{i'}^n \otimes M_i^n) \\ &\quad + \sum_{k < i < j < n} (\lambda'_{ij} M_i^j \otimes M_{i'}^{j'} + q^2 \lambda_{ij} M_{i'}^{j'} \otimes M_i^j) \end{aligned} \quad (9.3)$$

We have introduced the notation

$$i' := 2n - i, \text{ for } 0 < i < n. \quad (9.4)$$

The relations of the corresponding quantum (anti-) plane are the same as in (2.3, 2.4) when $i + j \neq 2n$ and, for $j < n$:

$$\begin{aligned} [x^j, x^{j'}]_{1/q^2} &= q^{-2} \sum_{i < j} \lambda_{ij} x^{i'} x^i = - \sum_{i < j} \lambda'_{ij} x^i x^{i'}, \\ [\theta^j, \theta^{j'}] &= - \sum_{i < j} \lambda_{ij} \theta^{i'} \theta^i = - \sum_{i < j} \lambda'_{ij} \theta^i \theta^{i'}, \\ \theta^n \theta^n &= - \sum_{i, n} \mu'_i \theta^{i'} \theta^i, \\ [x^j, \theta^{j'}]_{1/q^2} &= q^{-2} \sum_{i < j} \lambda_{ij} \theta^{i'} x^i + (q^{-2} - 1) \theta^j x^{j'}, \\ [\theta^j, x^{j'}] &= -q^{-2} \sum_{i < j} \lambda'_{ij} \theta^i x^{i'}, \\ [\theta^n, x^{j'}] &= - \sum_{i < j} \lambda'_{ij} \theta^i x^{i'}, \\ [\theta^n, x^n]_{q^2} &= -q^2 \sum_{i < j} (\mu'_i \theta^i x^{i'} + \mu_i \theta^{i'} x^i). \end{aligned} \quad (9.5)$$

Here $[A, B]_q := AB - qBA$. The braid relation is easily imposed by invoking Theorem 3, with the result that the coefficients μ, μ', λ and λ' must satisfy the conditions

$$\begin{aligned} \mu'_i &= -q^{2(i-n)} \mu_i, & i < n, \\ \mu_j \lambda_{ij} &= (1 - q^2) q^{2(i-j)} \mu_i, & i < j < n, \\ \mu_j \lambda'_{ij} &= (q^2 - 1) \mu_i, & i < j < n. \end{aligned} \quad (9.6)$$

The parameters $(\mu_i), i = 1, 2, \dots, n$, remain arbitrary except that $\mu_i = 0$ implies $\mu_{i-1} = 0$,

Esoteric quantum $gl(2n - 1)$ (in the Woronowicz picture) is generated by the matrix elements z_i^j of a matrix T , of dimension $2n - 1$, with relations given by

$$P(T \otimes T) = (T \otimes T)P. \quad (9.7)$$

These relations are complicated and we refer to [9].

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